Continuous-time stochastic models

Modern Techniques in Modelling

LONDON SCHOOL of HYGIENE &TROPICAL MEDICINE

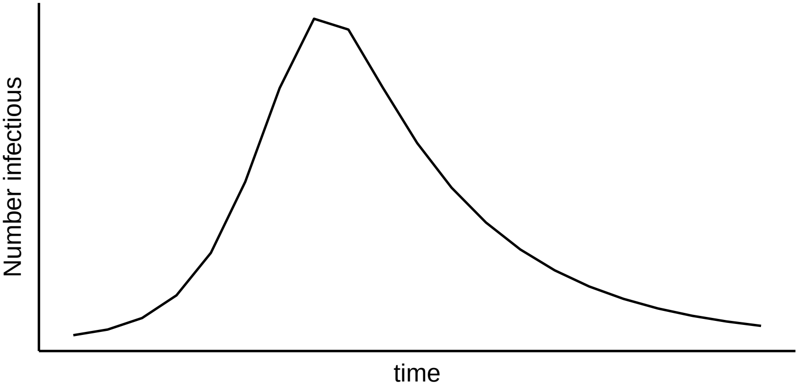
Introduction

Overview

- Introduce continuous-time stochastic models (~20 minutes)
- Implement the Gillespie algorithm and analyse stochastic model output (~60 minutes)
- Implement a stochastic model with the `adaptivetau` package (~20 minutes)
- Discussion and concluding remarks (~20 minutes)

Deterministic models

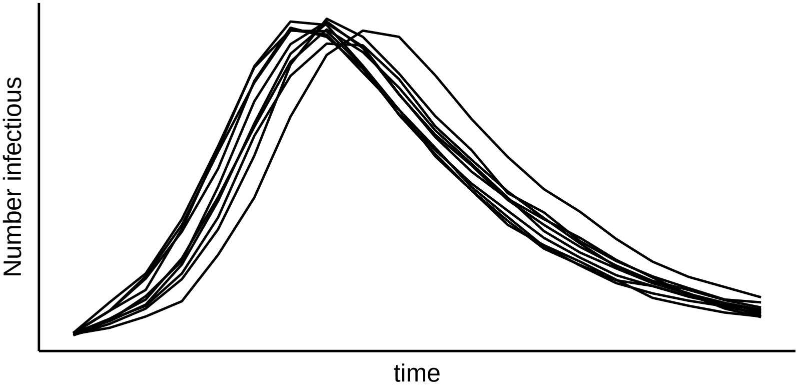
SIR model with $I_0 = 10, \beta = 1.3, \gamma = 0.3$



One set of parameters $\rightarrow$ one trajectory

Stochastic models

SIR model with $I_0 = 10, \beta = 1.3, \gamma = 0.3$



One set of parameters $\rightarrow$ many trajectories

Types of model

- *discrete* vs *continuous* time
- *compartment*- vs *individual*-based
- *deterministic* vs *stochastic* dynamics

Session 3: Discrete-time deterministic models

- *discrete* vs *continuous* time
- *compartment*- vs *individual*-based
- *deterministic* vs *stochastic* dynamics

$$
\begin{aligned}
S(t+1) &= S(t) - \beta S(t) I(t) \\
I(t+1) &= I(t) + \beta S(t) I(t) - \gamma I(t) \\
R(t+1) &= R(t) + \gamma I(t)
\end{aligned}
$$

Session 3: Discrete-time deterministic models

- **discrete** vs ~~continuous~~ time
- **compartment-** vs ~~individual-~~based
- **deterministic** vs ~~stochastic~~ dynamics

$$\begin{aligned} S(t+1) &= S(t) - \beta S(t)I(t) \\ I(t+1) &= I(t) + \beta S(t)I(t) - \gamma I(t) \\ R(t+1) &= R(t) + \gamma I(t) \end{aligned}$$

Session 4: Ordinary differential equations

- *discrete* vs *continuous* time
- *compartment*- vs *individual*-based
- *deterministic* vs *stochastic* dynamics

$$\begin{aligned} dS/dt &= -\beta SI/N \\ dI/dt &= \beta SI/N - \gamma I \\ dR/dt &= \gamma I \end{aligned}$$

Session 4: Ordinary differential equations

- ~~discrete~~ vs **continuous** time
- **compartment-** vs ~~individual-~~based
- **deterministic** vs ~~stochastic~~ dynamics

$$
\begin{aligned}
dS/dt &= -\beta SI/N \\
dI/dt &= \beta SI/N - \gamma I \\
dR/dt &= \gamma I
\end{aligned}
$$

- *discrete* vs *continuous* time
- *compartment-* vs *individual*-based
- *deterministic* vs *stochastic* dynamics

```
For each ts from 1 to T {
  lambda <- beta * I/N
  For each i from 1 to N {
    If individual i is susceptible:
      with prob 1-exp(-lambda· t) make infected.
    Else-if individual i is infected:
      with prob 1-exp(-gamma· t) make susceptible.
  }
  Record population state
}
```

Session 8: Stochastic individual-based models

- **discrete** vs ~~continuous~~ time
- ~~compartment-~~ vs **individual-**based
- ~~deterministic~~ vs **stochastic** dynamics

```
For each ts from 1 to T {
  lambda <- beta * I/N
  For each i from 1 to N {
    If individual i is susceptible:
      with prob 1-exp(-lambda· t) make infected.
    Else-if individual i is infected:
      with prob 1-exp(-gamma· t) make susceptible.
  }
  Record population state
}
```

Continuous-time stochastic models

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Continuous-time stochastic models

- ~~discrete~~ vs **continuous** time
- *compartment*- vs *individual*-based
- ~~deterministic~~ vs **stochastic** dynamics

Stochastic differential equations (SDEs)

- ~~discrete~~ vs **continuous** time
- **compartment-** vs ~~individual-~~based
- ~~deterministic~~ vs **stochastic** dynamics

$$
\begin{aligned}
dS/dt &= -\beta SI/N - \sqrt{\beta SI/N}dW_1 \\
dI/dt &= \beta SI/N - \gamma I + \sqrt{\beta SI/N}dW_1 - \sqrt{\gamma I}dW_2 \\
dR/dt &= \gamma I + \sqrt{\gamma I}dW_2
\end{aligned}
$$

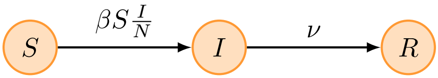
(not covered in this course)

Continuous-time discrete stochastic models

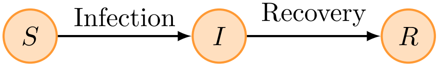
- ~~discrete~~ vs **continuous** time
- ~~compartment-~~ vs **individual-**based
- ~~deterministic~~ vs **stochastic** dynamics

We model these as a so-called **continuous-time Markov chains**.

Event-based view



Event-based view



- **infection**: $(S, I, R) \to (S-1, I+1, R)$ with rate $\beta SI/N$
- **recovery**: $(S, I, R) \to (S, I-1, R+1)$ with rate γI

Moving to continuous time

Discrete time

```
for (ts in 1:steps) {
  update all compartments
}
```

(see Session 8: Stochastic individual-based models)

Continuous time

```
while (time < finaltime) {
  advance time and record next event
}
```

(this session)

Gillespie algorithm

Repeat until end time:

1. Calculate rates of all possible events

```
rates <- c(
  infection =  beta * S * I / N,
  recovery =  gamma * I
)
```

2. Determine time of next event

```
rexp(n =  1, rate = sum(rates))
```

3. Determine which event happens

```
sample(x = length(rates), size = 1, prob = rates)
```

and update system state according to event.

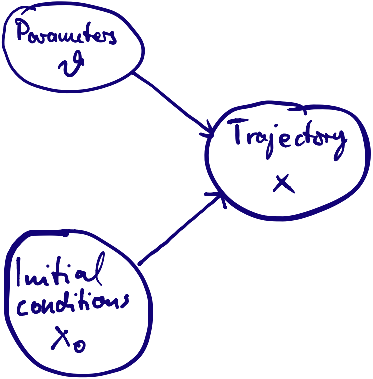
Now, put it in R

Practical structure

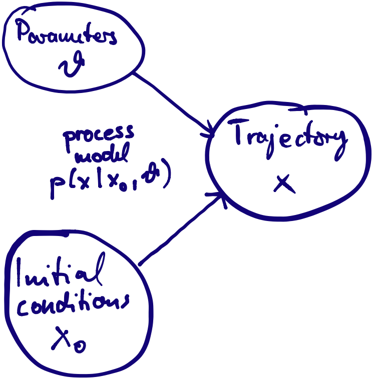
- Part 1: Stochastic simulations using the Gillespie algorithm
- Part 2: A faster alternative: the `adaptivetau` package

Representing uncertainty

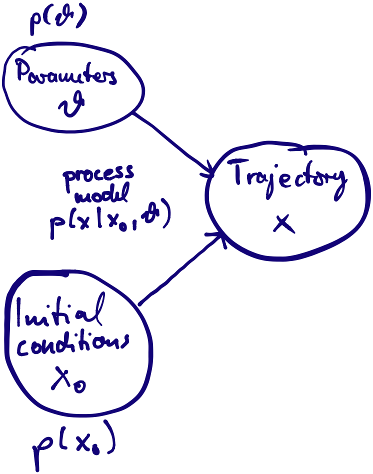
The deterministic view



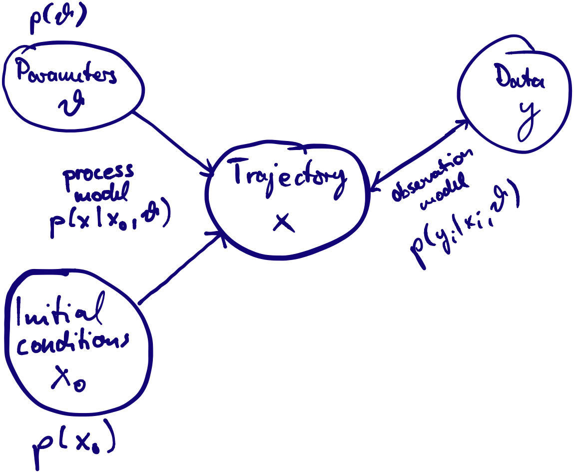
The stochastic view



Other types of uncertainty



Linking models to data



See LSHTM short course on Model Fitting and Inference for Infectious Disease Dynamics.

Further reading

- L.J.S. Allen (2017). A primer on stochastic epidemic models: Formulation, numerical simulation, and analysis. Infectious Disease Modelling, 2(2):128–142. https://doi.org/10.1016/j.idm.2017.03.001
- M.J. Keeling, P. Rohani (2017). Modeling Infectious Diseases in Humans and Animals. Princeton University Press.
- D.T. Gillespie (1976). A general method for numerically simulating the stochastic time evolution of coupled chemical reactions. J Comput Phys, 22(4):403–434, 1976. ISSN 0021-9991. https://doi.org/10.1016/0021-9991(76)90041-3
- Y. Cao, D.T. Gillespie, and L.R. Petzold (2007). Adaptive explicit-implicit tau-leaping method with automatic tau selection. J Chem Phys, 126(22):224101 URL https://doi.org/10.1063/1.2745299
- A.A. King, M. Domenech de Cellès, F.M.G. Magpantay and Pejman Rohani (2015). Avoidable errors in the modelling of outbreaks of emerging pathogens, with special reference to Ebola. Proc Roy Soc B 282(1806). https://doi.org/10.1098/rspb.2015.0347